

Two-dimensional metric-affine gravity

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Abstract

There is a number of completely integrable gravity theories in two dimensions. We study the metric-affine approach on a 2-dimensional spacetime and display a new integrable model. Its properties are described and compared with the known results of Poincaré gauge gravity.

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I. INTRODUCTION

Although it is well known that the Hilbert-Einstein Lagrangian yields the trivial dynamics in 2 dimensions, 2-dimensional gravitational theory has attracted considerable attention recently. In particular, the extension of the spacetime geometry by including nontrivial torsion has encompassed a class of interesting models with rich mathematical and physical contents [1–5]. For an overview of the relevant results and a more exhaustive list of publications, we refer to [6,7]. These models are remarkable for at least two reasons. Namely, they provide a convenient tool for the study of sting-motivated dilaton gravity which can be formulated as an effective Poincaré gauge model. And moreover, they represent several examples of physically meaningful completely integrable models.

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The purpose of this paper is to generalize the 2-dimensional gravity models by taking into account all possible post-Riemannian geometrical structures. In practical terms, this means to take into consideration both the torsion and the nonmetricity, thus extending the earlier results [1–5] which were confined only to metric-compatible connections with torsion. The general framework for metric-affine gravity (MAG) is firmly established on the basis of the gauge approach for the general affine group in a spacetime of any dimension [8]. The specific applications to string and dilaton gravity were discussed in [9], for example. The first (to our knowledge) 2-dimensional gravity model with nonmetricity [10], however, was constructed in terms of a nonmetric geometry with vanishing torsion. It is thus of interest to study the case of the most general metric-affine 2-dimensional spacetime with both torsion and nonmetricity being nontrivial. Technically, it seems natural to start with the quadratic Poincaré gauge model and to extend it by including the new quadratic nonmetricity term in the Lagrangian. We then will benefit from the techniques developed previously for the Poincaré gauge case [5,6]. Our main result is the demonstration of the complete integrability of the obtained model in vacuum.

The structure of the paper is as follows. In Sec. II we briefly summarize the basics of the MAG approach, with special attention to the irreducible decomposition of the geometric objects in 2 dimensions. Sec. III describes how the spacetime interval can be constructed with the help of the torsion, whereas Sec. IV is devoted to the formulation of the dynamical scheme of the model. The complete integrability in vacuum is demonstrated in Sec. V. Finally, in Sec VI we discuss the results obtained and outline the open problems.

Our basic notations and conventions are those of [8]. In particular, the signature of the 2-dimensional metric is assumed to be $(-, +)$. Spacetime coordinates are labeled by Latin indices, $i, j, \dots = 0, 1$ (for example, dx^i), whereas Greek indices, $\alpha, \beta, \dots = 0, 1$, label the local frame components (for example, the coframe 1-form ϑ^α). Along with the coframe 1-forms ϑ^α , we will use the so called η -basis of the dual coframes. Namely, we define the Hodge dual such that $\eta := *1$ is the volume 2-form. Furthermore, denoting by e_α the vector frame, we have the 1-form $\eta_\alpha := e_\alpha \rfloor \eta = *\vartheta_\alpha$, and the 0-form $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = *(\vartheta_\alpha \wedge \vartheta_\beta)$.

The last expression represents the 2-dimensional totally antisymmetric Levi-Civita tensor.

II. METRIC-AFFINE APPROACH

In metric-affine gravity (MAG), the gravitational field is described by the coframe one-form ϑ^α , the linear connection one-form Γ_α^β and the metric $g_{\alpha\beta}$. The first two variables are considered to be the gauge potentials of the gravitational field corresponding, respectively, to the translation group and the general linear group acting in the tangent space at each point of spacetime. The gravitational field strengths are given by the torsion two-form $T^\alpha := D\vartheta^\alpha$, the curvature two-form $R_\alpha^\beta := d\Gamma_\alpha^\beta - \Gamma_\alpha^\gamma \wedge \Gamma_\gamma^\beta$, and the nonmetricity 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$. The frame $e_\alpha = e^i_\alpha \partial_i$ is dual to the coframe $\vartheta^\beta = e_j^\beta dx^j$, i.e., $e_\alpha \lrcorner \vartheta^\beta = e^i_\alpha e_i^\beta = \delta_\alpha^\beta$. The spacetime manifold M is equipped with a line element

$$ds^2 = g_{ij} dx^i \otimes dx^j = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta. \quad (1)$$

More details about the MAG approach in any dimension can be found in [8].

Before we proceed with the analysis of the metric-affine gravity model in two dimensions, it is instructive to understand the structure of the basic geometric objects. Here we describe their independent components and irreducible parts.

In two dimensions, the *torsion* has 2 components. It reduces to its vector trace 1-form (second irreducible piece [8])

$$T^\alpha = \vartheta^\alpha \wedge T, \quad T := e_\alpha \lrcorner T^\alpha. \quad (2)$$

The 2-dimensional *nonmetricity* has 6 independent components and it decomposes into 3 irreducible parts: $Q_{\alpha\beta} = {}^{(1)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}$, with

$${}^{(3)}Q_{\alpha\beta} := \vartheta_{(\alpha} e_{\beta)} \lrcorner \Lambda - \frac{1}{2} g_{\alpha\beta} \Lambda, \quad (3)$$

$${}^{(4)}Q_{\alpha\beta} := g_{\alpha\beta} Q, \quad (4)$$

$${}^{(1)}Q_{\alpha\beta} := Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta}. \quad (5)$$

Here the shear covector part of the nonmetricity and the Weyl covector are, respectively,

$$\Lambda := \vartheta^\alpha e^\beta \rfloor \mathcal{Q}_{\alpha\beta}, \quad Q := \frac{1}{2} g^{\alpha\beta} Q_{\alpha\beta}, \quad (6)$$

where $\mathcal{Q}_{\alpha\beta} = Q_{\alpha\beta} - Q g_{\alpha\beta}$ is the traceless piece of the nonmetricity. For the spacetime dimension greater than 2, the nonmetricity has also a second irreducible piece [8] which vanishes identically ${}^{(2)}Q_{\alpha\beta} = 0$ in 2 dimensions.

The three 1-forms of torsion and nonmetricity (T, Q, Λ) form the so-called *triplet* which plays a significant role in the MAG theories in 4 dimensions [11]. As we will see, the triplet of 1-forms is important also for understanding of the 2-dimensional MAG.

The *curvature* 2-form has 4 components in 2 dimensions and it decomposes into the 3 irreducible pieces:

$$R_{\alpha\beta} = W_{\alpha\beta} + \mathcal{Z}_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} Z. \quad (7)$$

Here the skew-symmetric part $W_{\alpha\beta} := R_{[\alpha\beta]}$ is the direct generalization of the Riemann-Cartan curvature, whereas the symmetric part $Z_{\alpha\beta} := R_{(\alpha\beta)} = \frac{1}{2} DQ_{\alpha\beta}$ is only nontrivial in presence of the nonmetricity. The skew-symmetric part is irreducible in 2 dimensions. It has one independent component and it can be expressed in terms of the curvature scalar $R := e_\alpha \rfloor e_\beta \rfloor R^{\alpha\beta}$:

$$W^{\alpha\beta} = -\frac{1}{2} R \vartheta^\alpha \wedge \vartheta^\beta. \quad (8)$$

However the symmetric part is more nontrivial. It has 3 components and it can be decomposed into the trace 2-form $Z := g^{\alpha\beta} Z_{\alpha\beta}$ (1 component) and the traceless part $\mathcal{Z}_{\alpha\beta} := Z_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} Z$ (2 components).

The coframe ϑ^α is not covariantly constant in a general MAG spacetime. Similarly, the covariant derivatives of the η -objects are also non-vanishing. In particular, in 2 dimensions, we find explicitly:

$$D\eta^\alpha = \eta e^\alpha \rfloor \Lambda + \eta^{\alpha\beta} T_\beta, \quad (9)$$

$$D\eta^\alpha{}_\beta = {}^* \left({}^{(1)}Q^\alpha{}_\beta - {}^{(3)}Q^\alpha{}_\beta \right). \quad (10)$$

We will use these identities in the analysis of the MAG field equations.

III. TORSION AND LINE ELEMENT OF SPACETIME

In our demonstration of the complete integrability of the 2-dimensional MAG model, we will use the technical tools developed earlier for the Poincaré gauge theory [5,6]. Namely, in 2 dimensions, one can construct the spacetime interval from a non-degenerate torsion. Here we briefly summarize the corresponding definitions and results.

In 2 dimensions, the two independent components of the torsion can be described in terms of the vector-valued torsion zero-form t^α defined via the Hodge dual

$$t^\alpha := {}^*T^\alpha. \quad (11)$$

Then the torsion 2-form is recovered as

$$T^\alpha = -t^\alpha \eta. \quad (12)$$

We call the manifold M a *non-degenerate metric-affine spacetime* when the torsion square is not identically zero, i.e. $t^2 := t_\alpha t^\alpha \neq 0$. Then we can write a coframe as [5,6]:

$$\vartheta^\alpha = -\frac{1}{t^2} \left(T \eta^{\alpha\beta} t_\beta + {}^*T t^\alpha \right). \quad (13)$$

In other words, the torsion 1-form T and its dual *T specify a coframe with respect to which one can expand all the 2D geometrical objects. Such a coframe is non-degenerate when $t^2 \neq 0$, hence the terminology. The volume 2-form can be calculated as an exterior square of the torsion 1-form T :

$$\eta = \frac{1}{2} \eta_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta = \frac{1}{t^2} {}^*T \wedge T. \quad (14)$$

Defining a coframe of a 2-dimensional spacetime in terms of the torsion 1-form is an important step in the study of the integrability of the MAG model.

IV. GRAVITATIONAL FIELD EQUATIONS

We are now ready to formulate the dynamics of the MAG model in two dimensions. Let us consider the Lagrangian 2-form:

$$V(\vartheta^\alpha, T^\alpha, R_{\alpha}{}^\beta, Q_{\alpha\beta}) = \left(\frac{1}{2}R - \frac{b}{4}R^2 - \lambda \right) \eta - \frac{a_1}{2} T^\alpha \wedge {}^*T_\alpha - \frac{a_2}{2} Q^{\alpha\beta} \wedge {}^*Q_{\alpha\beta}. \quad (15)$$

Here a_1, a_2, b and λ (cosmological term) are the coupling constants. Recall that the Lagrangian of the Poincaré gauge model in 2 dimensions contained only the linear and quadratic contractions of curvature and torsion – the first and the second terms on the right-hand side of (15). In order to investigate the influence of the nonmetricity, we have only minimally modified the Poincaré Lagrangian by adding the last term quadratic in the nonmetricity. However, it is necessary also to note that the curvature scalar R is now qualitatively different since it depends on the general linear connection and not on the Lorentz connection alone.

The gravitational field equations are derived from the total Lagrangian $V + L_{\text{mat}}$ by independent variations with respect to the metric $g_{\alpha\beta}$, the 1-form ϑ^α (coframe) and the 1-form $\Gamma_\beta{}^\alpha$ (connection). The corresponding so-called *zeroth*, *first* and *second* field equations read

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \quad (16)$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \quad (17)$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta. \quad (18)$$

The source terms in the right-hand sides are defined as the derivatives of the matter Lagrangian: $\sigma^{\alpha\beta} := 2\delta L_{\text{mat}}/\delta g_{\alpha\beta}$ (the metrical energy-momentum 2-form), $\Sigma_\alpha := \delta L_{\text{mat}}/\delta \vartheta^\alpha$ (the canonical energy-momentum 1-form), $\Delta^\alpha{}_\beta := \delta L_{\text{mat}}/\delta \Gamma_\alpha{}^\beta$ (the canonical hypermomentum 1-form). The gauge field momenta appearing in the left-hand sides of the field equations are given by

$$M^{\alpha\beta} := -2 \frac{\partial V}{\partial Q_{\alpha\beta}} = 2a_2 {}^*Q^{\alpha\beta}, \quad (19)$$

$$H_\alpha := - \frac{\partial V}{\partial T^\alpha} = a_1 t_\alpha, \quad (20)$$

$$H^\alpha{}_\beta := - \frac{\partial V}{\partial R_{\alpha}{}^\beta} = \frac{1}{2} \eta^\alpha{}_\beta (1 - bR). \quad (21)$$

Finally, the energy-momenta and the hypermomentum of the gravitational field are described by $m^{\alpha\beta} = 2\partial V/\partial g_{\alpha\beta}$ and

$$E_\alpha = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) H^\beta{}_\gamma + \frac{1}{2} (e_\alpha \rfloor Q_{\beta\gamma}) \wedge M^{\beta\gamma}, \quad (22)$$

$$E^\alpha{}_\beta = -\vartheta^\alpha \wedge H_\beta - M^\alpha{}_\beta. \quad (23)$$

One can show quite generally (in any dimension) [8] that the zeroth equation (16) is redundant: it is a consequence of (17), (18) and of the Noether identities. Accordingly, we will consider the system of the first and second field equations (17) and (18) which determine completely the dynamics of the gravitational field.

V. GENERAL VACUUM SOLUTION

We will now specialize to the vacuum case when the matter sources are absent, $\Sigma_\alpha = 0$, $\Delta^\alpha{}_\beta = 0$. As a preliminary remark, we notice that if we put the nonmetricity equal zero at this stage, $Q_{\alpha\beta} = 0$, we will not recover the results of the Poincaré gauge model. Similarly, the limit of $a_2 = 0$ does not yield the old results despite the fact that the gravitational Lagrangian then formally coincides with that of the 2-dimensional Poincaré gauge theory. It is important to realize that the MAG dynamics is very different from the Poincaré gauge case, in particular, the number of the field equations is now greater.

A. Second field equation

Substituting (19)-(21) into the second field equation (18), we find in vacuum:

$$\frac{1}{2} D [\eta^\alpha{}_\beta (1 - b R)] + a_1 \vartheta^\alpha t_\beta + 2a_2 {}^*Q^\alpha{}_\beta = 0. \quad (24)$$

Taking the trace, we obtain the relation between the torsion and the Weyl 1-forms:

$$Q = \frac{a_1}{4a_2} T. \quad (25)$$

Now, using the identity (10), we decompose (24) into the symmetric and antisymmetric parts:

$$\frac{1}{2} (1 - b R)^* \left({}^{(1)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} \right) + a_1^* \left(\vartheta_{(\alpha} e_{\beta)} \rfloor T - g_{\alpha\beta} T \right) + 2a_2^* Q_{\alpha\beta} = 0, \quad (26)$$

$$-\frac{b}{2} \eta_{\alpha\beta} dR + a_1 \vartheta_{[\alpha} t_{\beta]} = 0. \quad (27)$$

The symmetric part (26) yields

$${}^{(1)}Q_{\alpha\beta} = 0, \quad \Lambda = \frac{2a_1}{1 - bR - 4a_2} T. \quad (28)$$

As a result, we discover at the end the *triplet* structure of the torsion-nonmetricity sector when the only nontrivial pieces of the torsion and the nonmetricity remain the three 1-forms which are proportional to each other: $\Lambda \sim Q \sim T$.

Finally, the antisymmetric equation (27) yields

$$T = -\frac{b}{a_1} dR. \quad (29)$$

This is completely analogous to the corresponding result of the Poincaré gauge model [5,6].

B. First field equation

We begin the analysis of the first MAG field equation (17) by noticing that substitution of (15), (20), (21) and (19) in (22) yields

$$E_\alpha = -\tilde{\mathcal{V}} \eta_\alpha + \frac{a_2}{2} \left({}^*Q^{\beta\gamma} e_\alpha \rfloor Q_{\beta\gamma} + Q_{\beta\gamma} e_\alpha \rfloor {}^*Q^{\beta\gamma} \right). \quad (30)$$

Here $\tilde{\mathcal{V}} = a_1 t^2/2 - bR^2/4 + \lambda$. Obviously, the above 1-form has the properties:

$$\eta^\alpha \wedge E_\alpha = 0, \quad \vartheta^\alpha \wedge E_\alpha = -\tilde{\mathcal{V}} \eta. \quad (31)$$

Using the results of the previous subsection, we then find the first MAG equation explicitly:

$$a_1 Dt_\alpha = -\tilde{\mathcal{V}} \eta_\alpha + a_2 \left({}^*Q e_\alpha \rfloor Q - Q \eta_{\alpha\beta} e^\beta \rfloor Q \right). \quad (32)$$

This equation determines the components of the torsion vector. Recall, however, that ultimately we are interested in the spacetime interval (1) which is expressed in terms of the coframe (13). Thus, we need to find the 1-form T and its dual *T together with the quadratic

invariant of the torsion $t^2 = t_\alpha t^\alpha$. This is achieved by contracting (32) with η^α , ϑ^α , and t^α , respectively.

Contracting (32) with η^α and taking into account (31), we find $dT = 0$. Note that we need the identity (9) and the relation (28) for this. The result is consistent with the earlier explicit formula (29).

Contracting (32) with ϑ^α , we obtain

$$a_1 d^*T = \left(a_1 t^2 - 2\tilde{\mathcal{V}}\right) \eta. \quad (33)$$

Finally, contracting (32) with t^α , we get the differential equation for the function t^2 :

$$\frac{a_1}{2} dt^2 = \tilde{\mathcal{V}} T + \frac{a_1}{4} t^2 (Q + \Lambda). \quad (34)$$

Substituting (25), (28) and (29) into (34), we obtain the first-order ordinary differential equation, the integration of which yields t^2 as a function of the curvature scalar:

$$-t^2 = \frac{a^2 \rho}{2a_1^2 b} \left\{ \frac{[4b\lambda - (1 - 4a_2)2]}{a} e^{-\rho} \text{Ei}(\rho) - a\rho + a - 2(1 - 4a_2) + c_0 e^{-\rho} \right\}. \quad (35)$$

Here $\frac{1}{a} := \frac{1}{a_1} + \frac{1}{8a_2}$, and $\text{Ei}(\rho)$ is the exponential integral function of the variable

$$\rho = \frac{bR + 4a_2 - 1}{a}. \quad (36)$$

The meaning of the integration constant c_0 will be clarified in the next subsection.

C. Spacetime geometry

In order to find the spacetime interval, we will proceed along the same line as the in [5,6]. Namely, as we see from (29), the torsion 1-form T plays the role of the first leg of a zweibein and we can interpret R as one of the local spacetime coordinates. The form of the solution (35) suggests, furthermore, that it will be more convenient to replace R with ρ using the linear transformation (36).

The second leg of a zweibein will be then described by the dual torsion 1-form *T . Then following [5,6] we introduce the second local coordinate ζ and find in this way the general ansatz

$${}^*T = B(\zeta, \rho) d\zeta. \quad (37)$$

The unknown function $B(\zeta, \rho)$ is determined as follows. Substituting (37) into (33) and using (14), we find the differential equation:

$$\frac{a_1}{a} \partial_\rho B = (a_1 t^2 - 2\tilde{\mathcal{V}}) \frac{B}{a_1 t^2}. \quad (38)$$

Combining this with (34), we obtain the solution:

$$B = \frac{-t^2 e^\rho}{\rho} B_0, \quad (39)$$

with an arbitrary $B_0 = B_0(\zeta)$.

Thus, we have completely determined the coframe (13) in terms of the torsion 1-form (29), the dual torsion (37) and the quadratic torsion invariant (35). The spacetime interval (1) is then recovered straightforwardly as

$$ds^2 = - \frac{(-t^2) e^{2\rho}}{\rho^2} B_0^2 d\zeta^2 + \frac{a^2}{a_1^2} \frac{d\rho^2}{(-t^2)}. \quad (40)$$

Without loss of generality, we can evidently put $B_0 = 1$ by a redefinition of the local coordinate ζ .

In order to reveal the meaning of the integration constant c_0 in (35), we notice that $\xi = \partial_\zeta$ is the Killing vector field for the line element (40). As a result, we can verify that the energy 1-form

$$\varepsilon = \xi^\alpha \Sigma_\alpha \quad (41)$$

is strongly conserved, $d\varepsilon = 0$. The first MAG field equation (17) yields

$$\varepsilon = \frac{a_1 e^\rho}{2\rho} \left[d(t^2) + (1 - 1/\rho) t^2 d\rho + \frac{a^2}{a_1^2} (2\lambda - bR^2/2) d\rho \right]. \quad (42)$$

Using this, we can verify that $\varepsilon = dM$ with

$$M = \frac{a_1 t^2 e^\rho}{2\rho} + \frac{a^2 e^\rho}{4a_1 b} \left\{ \frac{[4b\lambda - (1 - 4a_2)2]}{a} e^{-\rho} \text{Ei}(\rho) - a\rho + a - 2(1 - 4a_2) \right\}. \quad (43)$$

In vacuum, $\Sigma_\alpha = 0$ and hence $\varepsilon = dM = 0$. Thus, $M = M_0$ is constant for the vacuum solution obtained. Substituting (35) into (43), we find explicitly

$$c_0 = -\frac{4a_1b}{a^2} M_0. \quad (44)$$

By construction, M_0 represents the total mass of the configuration, cf. the discussion in [5].

D. Torsion-degenerate spacetime

Above, we have described a non-degenerate metric-affine spacetime, the geometric properties of which are totally determined by the 2-dimensional torsion. For completeness, we also need to analyse the case of degenerate torsion with $t^2 = 0$ everywhere. It is straightforward to see that the first and the second MAG field equations (17), (18) yield

$$T^\alpha = 0, \quad Q_{\alpha\beta} = 0, \quad R = 2\sqrt{\frac{\lambda}{b}}. \quad (45)$$

The resulting manifold is thus isometric to the 2-dimensional de Sitter space with vanishing torsion and nonmetricity. This result is analogous to the degenerate solutions of the Poincaré gauge model [1,4–6].

VI. DISCUSSION AND CONCLUSION

In this paper we have studied the quadratic MAG model (15) in 2-dimensional spacetime. We demonstrated that such a theory represents a new example of a completely integrable model of 2-dimensional gravity. This result was obtained by means of a direct extension of the general framework which was developed earlier for the case of the Poincaré gauge models. Not surprisingly, the form of the general vacuum solution resembles the solution of the quadratic Poincaré model. However, the old results are not recovered in the formal limit of the vanishing nonmetricity coupling constant $a_2 \rightarrow 0$. In particular, let us recall that it is possible to interpret the solutions of the Poincaré model as 2-dimensional black holes. In our current approach, the analysis of the possible black hole structure is related to the study

of zeros of the metric coefficient $g_{\zeta\zeta} = t^2 e^{2\rho}/\rho^2$ for the general solution (35). For certain values of the coupling constants (a_1, a_2, b, λ) , the resulting geometry may indeed display the black hole features similar to the black holes discovered in the quadratic Poincaré model. However, in general, the new solutions obtained are no black holes.

As compared to the Poincaré gauge case, qualitatively, the curvature and the torsion remain the basic elements of the theory. The scalar curvature acts as one of the local coordinates of the spacetime manifold, whereas the torsion defines a special coframe and thus provides the tool for constructing the spacetime interval. In addition, one of our primary goals was to study the specific role and place of the nonmetricity. Quite interestingly, it turns out that the non-Riemannian sector of the MAG model is represented by the triplet structure (T, Q, Λ) of the torsion and nonmetricity 1-forms. Such a triplet ansatz plays an important role in the 4-dimensional case [11]. The presence of the nonmetricity strongly modifies the vacuum solution. In particular, it introduces more singularities into the Riemannian geometry as compared to the effect of the torsion in the Poincaré model. The Riemannian curvature of the metric (40) can be easily computed from the corresponding Christoffel symbols and it reads explicitly

$$\tilde{R} = R - \frac{2a_1^2 t^2}{a^2 \rho^2} - \frac{2}{ab} [4b\lambda - (bR)^2] (1 - 1/\rho). \quad (46)$$

One may notice that the Lagrangian (15) does not describe the most general MAG model in two dimensions. Indeed, since the nonmetricity has three irreducible parts, we can extend (15) by including two more quadratic $(Q \cdot Q)$ terms, and furthermore to add a nonmetricity times torsion term of the form $(Q \cdot T)$. Such an extension, however, does not change our main result: The dynamics of the MAG fields remains qualitatively the same as in the minimal model (15), and the generalized model is also completely integrable. However, the inclusion of the nontrivial matter sources (scalar or spinor) as well as further extension of the model (15) by adding the quadratic terms of $Z_{\alpha\beta}$ destroys the integrability, in general.

The last but not least remark is about the origin of integrability property of the MAG model. There exists a systematic way of embedding the 2-dimensional (dilaton and Poincaré

gauge) gravity into the class of so-called Poisson-Sigma models [3,7]. The corresponding theoretical machinery provides an effective tool for demonstrating their complete integrability. A preliminary analysis shows that a similar interpretation of the new MAG model as a Poisson-Sigma model seems also to be possible.

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